習題 7-2

1. (1) (a)繞 X 軸旋轉的體積

$$V=\pi\int_0^2 x^2dx=\pi\cdot\frac{1}{3}x^3\Big|_0^2=\pi\cdot\frac{1}{3}(8-0)=\frac{8}{3}\pi$$

(b) 繞 Y 軸旋轉的體積

$$V=2\pi\int_0^2 x\cdot xdx=2\pi\int_0^2 x^2dx=2\pi\cdot\frac{1}{3}x^3\Big|_0^2=2\pi\cdot\frac{1}{3}(8-0)=\frac{16}{3}\pi$$

(2) (a)繞 X 軸旋轉的體積

$$V=\pi\int_0^8(2\sqrt{x})^2dx=4\pi\int_0^8 xdx=4\pi\cdot\frac{1}{2}x^2\Big|_0^8=2\pi(256-0)=512\pi$$

(b) 繞 Y 軸旋轉的體積

$$V=2\pi\int_0^8 x\cdot 2\sqrt{x}dx=4\pi\int_0^8 x^{\frac{3}{2}}dx=4\pi\cdot\frac{2}{5}x^{\frac{5}{2}}\Big|_0^8=\frac{8\pi}{5}(128\sqrt{2}-0)=\frac{1024\sqrt{2}}{5}\pi$$

(3) (a)繞 X 軸旋轉的體積

$$V=\pi\int_3^4(\sqrt{25-x^2})^2dx=\pi\int_3^4 25-x^2$$

$$=\pi[25x-\frac{1}{3}x^3\Big|_3^4]=\pi[25(4-3)-\frac{1}{3}(64-27)]=\frac{38}{3}\pi$$

(b) 繞 Y 軸旋轉的體積

$$V=2\pi\int_3^4 x\cdot\sqrt{25-x^2}dx$$

令 $u=25-x^2,\ du=-2xdx$

$$V=2\pi\int_3^4 x\cdot\sqrt{25-x^2}dx=-\pi\int_{16}^{9}\sqrt{u}du=-\pi\frac{2}{3}u^{\frac{3}{2}}\Big|_{16}^{9}=\frac{2\pi}{3}(64-27)=\frac{74}{3}\pi$$

(4) (a)繞 X 軸旋轉的體積

$$V=\pi\int_1^e(\ln x)^2dx$$

令

$u=(\ln x)^2 \qquad dv=dx$

$du=2(\ln x)\frac{1}{x}dx \qquad v=x$

$$\int(\ln x)^2dx=x(\ln x)^2-2\int\ln xdx$$

令

$$u=\ln x \qquad dv=dx$$

$$du=\frac{1}{x}dx \qquad v=x$$

$$\int \ln x\,dx = x\ln x-\int dx = x\ln x - x + C$$

所以

$$V=\pi\int_1^e(\ln x)^2dx=\pi[x(\ln x)^2-2(x\ln x-x)]_1^e=\pi(e-2)$$

(b) 繞 Y 軸旋轉的體積

$$V=2\pi\int_1^e x\cdot\ln x dx$$

令

$$u=\ln x \qquad dv=xdx$$

$$du=\frac{1}{x}dx \qquad v=\frac{1}{2}x^2$$

$$\int x\ln x\,dx=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x\,dx=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$$

所以

$$V=2\pi\int_1^e x\cdot\ln xdx=2\pi[\frac{1}{2}x^2\ln x-\frac{1}{4}x^2]_1^e=2\pi(\frac{1}{4}e^2+\frac{1}{4})=\frac{1}{2}\pi(e^2+1)$$

（5）(a)繞 X 軸旋轉的體積

$$V=\pi\int_0^{\pi}(\sin x)^2dx=\pi\int_0^{\pi}\frac{1-\cos 2x}{2}dx=\pi\,(\frac{1}{2}x-\frac{1}{4}\sin 2x)\Big|_0^{\pi}=\frac{1}{2}\pi^2$$

(b) 繞 Y 軸旋轉的體積

$$V=2\pi\int_0^{\pi}x\cdot\sin xdx$$

令

$$u=x \qquad dv=\sin xdx$$

$$du=dx \qquad v=-\cos x$$

所以

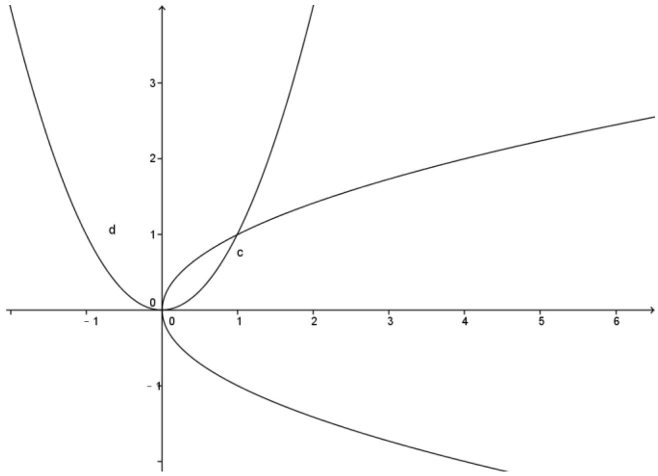
$$V=2\pi\int_0^{\pi}x\cdot\sin xdx=2\pi[-x\cos x\Big|_0^{\pi}+\int_0^{\pi}\cos xdx]=2\pi[\pi+\sin x\Big|_0^{\pi}]=2\pi^2$$

2. 可考慮 $y=b\sqrt{1-\frac{x^2}{a^2}},x\in[-a,a]$與 X 軸所為區域繞 X 軸旋轉的體積，所求為

$$V=\pi\int_{-a}^{a}y^2dx=\pi\int_{-a}^{a}b^2(1-\frac{x^2}{a^2})dx=b^2\pi(x-\frac{1}{3a^2}x^3\Big|_{-a}^{a})$$

$$= b^2\pi[\frac{2a}{3} - (-\frac{2a}{3})] = \frac{4}{3}ab^2\pi$$

3. 曲線 $y^2 = x(y = \sqrt{x}, y \geq 0)$ 與 $y = x^2$ 所圍區域的交點在 $(0,0)$, $(1,1)$ ，



所以 (1)繞 X 軸旋轉的體積

$$V = \pi\int_0^1 (\sqrt{x})^2 - (x^2)^2 dx = \pi\int_0^1 x - x^4 dx = \pi(\frac{1}{2}x^2 - \frac{1}{5}x^5 \Big|_0^1) = \frac{3\pi}{10}$$

(2) 繞 Y 軸旋轉的體積

$$V = 2\pi\int_0^1 x \cdot (\sqrt{x} - x^2)dx = 2\pi\int_0^1 x^{\frac{3}{2}} - x^3 dx = 2\pi(\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{4}x^4 \Big|_0^1) = \frac{3\pi}{10}$$

(3)繞直線 $x = 2$ 旋轉的體積

$$V = 2\pi\int_0^1 (2 - x) \cdot (\sqrt{x} - x^2)dx = 2\pi\int_0^1 2x^{\frac{1}{2}} - x^{\frac{3}{2}} - 2x^2 + x^3 dx$$

$$= 2\pi(\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^3 + \frac{1}{4}x^4 \Big|_0^1) = \frac{31\pi}{30}$$

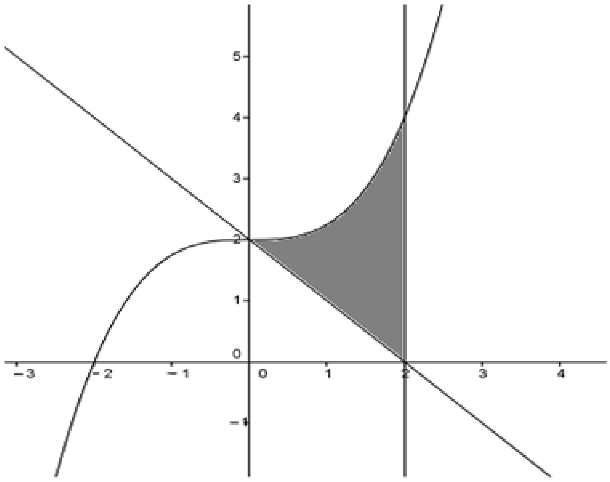
(4)繞直線 $y = -1$ 旋轉的體積

$$V = \pi\int_0^1 (\sqrt{x} - (-1))^2 - (x^2 - (-1))^2 dx$$

$$= \pi\int_0^1 x + 2\sqrt{x} - x^4 - 2x^2 dx$$

$$= \pi(\frac{1}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}} - \frac{1}{5}x^5 - \frac{2}{3}x^3 \Big|_0^1) = \frac{29\pi}{30}$$

4.



Sol：(1) 繞 Y 軸旋轉的體積

$$V = 2\pi\int_0^2 x\cdot[\frac{1}{4}x^3 + 2 - (2-x)]dx = 2\pi\int_0^2 \frac{1}{4}x^4 + x^2 dx$$

$$= 2\pi(\frac{1}{20}x^5 + \frac{1}{3}x^3\Big|_0^2) = \frac{128\pi}{15}$$

(2)繞直線 $x = -1$ 旋轉的體積

$$V = 2\pi\int_0^2 (x-(-1))\cdot[\frac{1}{4}x^3 + 2 - (2-x)]dx = 2\pi\int_0^2 (x+1)(\frac{1}{4}x^3 + x)dx$$

$$= 2\pi\int_0^2 \frac{1}{4}x^4 + \frac{1}{4}x^3 + x^2 + xdx$$

$$= 2\pi(\frac{1}{20}x^5 + \frac{1}{16}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2\Big|_0^2) = \frac{218\pi}{15}$$

5. 所求可考慮 $y = \sqrt{4-x^2}, x \in [-2,2]$ 與 X 軸所為區域繞 $x = 3$ 旋轉的體積的二倍

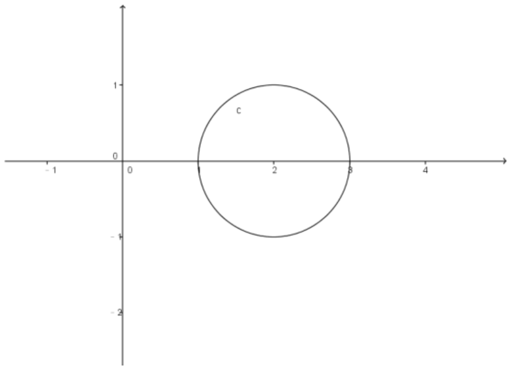
$$V = 2[2\pi\int_{-2}^2 (3-x)\cdot\sqrt{4-x^2}dx]$$

$$= 12\pi\int_{-2}^2 \sqrt{4-x^2}dx - 4\pi\int_{-2}^2 x\sqrt{4-x^2}dx$$

$$= 12\pi\cdot\frac{1}{2}\pi\cdot 2^2 + 0 \ = 24\pi^2$$

第一項積分值是半徑為 2 的半圓面積，第二項積分函數為奇函數積分值為 0。

6. 所求可考慮 $y = \sqrt{1-(x-2)^2}, x \in [1,3]$ 與 X 軸所為區域繞 Y 軸旋轉的體積的二倍



$$V = 2[2\pi\int_1^3 x\cdot\sqrt{1-(x-2)^2}dx]$$

令 $u = x-2,\ du = dx$

$$\begin{aligned}V &= 2[2\pi\int_1^3 x\cdot\sqrt{1-(x-2)^2}dx]\\ &= 4\pi\int_{-1}^1 (u+2)\sqrt{1-u^2}du\\ &= 4\pi\int_{-1}^1 u\sqrt{1-u^2}du + 8\pi\int_{-1}^1\sqrt{1-u^2}du\\ &= 0+8\pi\cdot\frac{1}{2}\pi\cdot 1^2 = 4\pi^2\end{aligned}$$

(第一項積分函數為奇函數積分值為 0，第二項是半徑為 1 的半圓面積。)

7. 底圓方程式為 $x^2+y^2=9$，對每個 x，$x\in[-3,3]$ 有對應的直徑 $2y$，所生成的正方形面積為 $A(x)=(2y)(2y)=4y^2=4(9-x^2)$，所以立體的體積為

$$V=\int_{-3}^3 4(9-x^2)dx = 4(9x-\frac{1}{3}x^3\Big|_{-3}^3)=144$$

習題 9-2

1. (1) $\displaystyle\lim_{(x,y)\to(3,2)}\frac{xy}{x^2-y^2}=\frac{3\cdot 2}{3^2-2^2}=\frac{6}{5}$

(2) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{\sin(5x^2+5y^2)}{x^2+y^2}=\lim_{r\to 0}\frac{\sin(5r^2)}{r^2}=\lim_{r\to 0}\frac{\sin(5r^2)}{5r^2}\cdot 5=1\cdot 5=5$

(3) 當沿直線 $x=0$ 逼近時

$$\lim_{(0,y)\to(0,0)}\frac{0y^3}{0^4+y^4}=\lim_{(0,y)\to(0,0)}0=0$$

當沿直線 $x=y$ 逼近時

$$\lim_{(y,y)\to(0,0)}\frac{y\cdot y^3}{y^4+y^4}=\lim_{(y,y)\to(0,0)}\frac{1}{2}=\frac{1}{2}$$

因 $0\neq\frac{1}{2}$，所以 $\lim\limits_{(x,y)\to(0,0)}\frac{xy^3}{x^4+y^4}$ 不存在。

(4) 當沿直線 $x=0$ 逼近時

$$\lim_{(0,y)\to(0,0)}\frac{0^2y}{0^4+y^2}=0$$

當沿曲線 $y=x^2$ 逼近時

$$\lim_{(x,x^2)\to(0,0)}\frac{x^2\cdot x^2}{x^4+(x^2)^2}=\lim_{(x,x^2)\to(0,0)}\frac{1}{2}=\frac{1}{2}$$

因 $0\neq\frac{1}{2}$，所以 $\lim\limits_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$ 不存在。

(5) $$\lim_{(x,y)\to(0,0)}\frac{4xy}{\sqrt{x^2+y^2}}=\lim_{r\to 0}\frac{4\cdot r\cos\theta\cdot r\sin\theta}{r}=\lim_{r\to 0}4r\cos\theta\sin\theta=0$$

(6) $$\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y^2}=\lim_{r\to 0}\frac{(r\cos\theta)^3+(r\sin\theta)^3}{r^2}=\lim_{r\to 0}r(\cos^3\theta+\sin^3\theta)=0$$

2. (1) $f(x,y)=\sqrt{x^2+y^2}$ 在 $\Re^2$ 上均連續，所以為連續函數。

(2) $f(x,y)=\ln(x^2+4y^2+16)$ 在 $\Re^2$ 上均連續，所以為連續函數。

(3) $f(x,y)=\sqrt{y}\cos\sqrt{x+y}$ 只在 $\{(x,y)|\,y\geq 0, x+y\geq 0\}$ 上連續，所以非連續函數。

(4) $f(x,y)$ 只在 $\{(x,y)|\,2x+y\neq 0\}$ 上連續，所以非連續函數。

3. (1) 因為 $\lim\limits_{(x,y)\to(0,0)}f(x,y)=\lim\limits_{(x,y)\to(0,0)}\frac{x^2+y^2}{|x|+|y|}=\lim\limits_{r\to 0}\frac{r^2}{|r\cos\theta|+|r\sin\theta|}$

$$=\lim_{r\to 0}\frac{r}{|\cos\theta|+|\sin\theta|}=0=f(0,0)$$

所以在 $(0,0)$ 連續。

(2) 當沿直線 $x=0$ 逼近時

$$\lim_{(0,y)\to(0,0)}f(0,y)=\lim_{(0,y)\to(0,0)}0=0$$

當沿直線 $y=x$ 逼近時

$$\lim_{(x,x)\to(0,0)}f(x,x)=\lim_{(x,x)\to(0,0)}\frac{1}{2}=\frac{1}{2}$$

因 $0\neq\frac{1}{2}$，所以 $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ 不存在，因此在 $(0,0)$ 不連續。。

(3) 因為 $\lim\limits_{(x,y)\to(0,0)}f(x,y)=\lim\limits_{(x,y)\to(0,0)}\frac{5x^2y^3}{(x^2+y^2)^2}$

$$=\lim_{r\to 0}\frac{5(r\cos\theta)^2(r\sin\theta)^3}{r^4}$$

$$=\lim_{r\to 0}5r\cos^2\theta\sin^3\theta=0=f(0,0)$$

所以在 $(0,0)$ 連續。

4. 因為 $\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2+y^2}$

$$=\lim_{r\to 0}\frac{(r\cos\theta)^2(r\sin\theta)^2}{r^2}=\lim_{r\to 0}r^2\cos^2\theta\sin^2\theta=0$$

所以 $f(0,0)=0$ 為所求

習題 9-3

1. $f_x=3x^2+14xy+3,\ f_y=7x^2+24y^2-2$

2. $\frac{\partial z}{\partial x}=\tan^{-1}\frac{y}{x}+x\frac{1}{1+\left(\frac{y}{x}\right)^2}(-\frac{y}{x^2})=\tan^{-1}\frac{y}{x}-\frac{xy}{x^2+y^2},$

$$\frac{\partial z}{\partial y}=x\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}=\frac{x^2}{x^2+y^2}$$

3.

$$f_x(x,y)=e^{-x}(-1)\sin(x+2y)+e^{-x}\cos(x+2y)$$

$$\therefore f_x(0,\frac{\pi}{4})=e^0(-1)\sin(\frac{\pi}{2})+e^0\cos(\frac{\pi}{2})=-1,$$

$$f_y(x,y)=e^{-x}\cos(x+2y)(2),$$

$$\Rightarrow f_y(0,\frac{\pi}{4})=0$$

4. 因為 $\frac{\partial f}{\partial x}=\frac{1}{2\sqrt{x^2+y^2+z^2}}2x=\frac{x}{\sqrt{x^2+y^2+z^2}}$

同理 $\frac{\partial f}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{\partial f}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}$

所以

$$(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2+(\frac{\partial f}{\partial z})^2$$

$$=(\frac{x}{\sqrt{x^2+y^2+z^2}})^2+(\frac{y}{\sqrt{x^2+y^2+z^2}})^2+(\frac{z}{\sqrt{x^2+y^2+z^2}})^2$$

$$=1$$

5. $\frac{\partial f}{\partial z}=(xy)^{\sin z}\ln(xy)\frac{\partial}{\partial z}\sin z=(xy)^{\sin z}\ln(xy)\cos z$

6. 因為 $\frac{\partial z}{\partial x}=\frac{1}{x^2+y^2+xy}(2x+y),\ \frac{\partial z}{\partial y}=\frac{1}{x^2+y^2+xy}(2y+x)$

所以 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\frac{2x+y}{x^2+y^2+xy}+y\frac{2y+x}{x^2+y^2+xy}=\frac{2(x^2+y^2+xy)}{x^2+y^2+xy}=2$

7. $\frac{\partial u}{\partial x}=1(y-z)(z-x)+(x-y)(y-z)(-1)$

$\frac{\partial u}{\partial y}=-1(y-z)(z-x)+(x-y)1(z-x)$

$\frac{\partial u}{\partial z}=(x-y)(-1)(z-x)+(x-y)(y-$

所以 $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

8. $\because \frac{\partial}{\partial R_3}(\frac{1}{R})=\frac{\partial}{\partial R_3}(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3})$

$\Rightarrow -\frac{1}{R^2}\frac{\partial}{\partial R_3}R=-\frac{1}{R_3{}^2}$

$\Rightarrow \frac{\partial R}{\partial R_3}=\frac{R^2}{R_3{}^2}=\frac{1}{R_3{}^2}\left(\frac{R_1R_2R_3}{R_2R_3+R_1R_3+R_1R_2}\right)^2=\left(\frac{R_1R_2}{R_2R_3+R_1R_3+R_1R_2}\right)^2$

9. $f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=\lim_{h\to 0}0=0$

$f_y(0,0)=\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=\lim_{k\to 0}0=0$

10. $f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h-0}{h}=\lim_{h\to 0}1=1$

$f_y(0,0)=\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{k-0}{k}=\lim_{k\to 0}1=1$

11. 因為 $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}=\frac{1}{2}\ln(x^2+y^2+z^2)$

所以 $f_x=\frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2x=\frac{x}{x^2+y^2+z^2}$

$f_{xx}=\frac{\partial}{\partial x}\frac{x}{x^2+y^2+z^2}=\frac{(x^2+y^2+z^2)\cdot 1-x\cdot 2x}{(x^2+y^2+z^2)^2}=\frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$

$f_{xxy}=\frac{\partial}{\partial y}\frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$

$=\frac{(x^2+y^2+z^2)^2\cdot(2y)-(-x^2+y^2+z^2)\cdot 2(x^2+y^2+z^2)\cdot 2y}{(x^2+y^2+z^2)^4}$

$=\frac{6x^2y-2y^3-2yz^2}{(x^2+y^2+z^2)^3},$

同理，由 y,z 的對稱性可得

$$f_{xxz}=\frac{6x^2z-2z^3-2y^2z}{(x^2+y^2+z^2)^3}$$

12. $f_x=3x^2+4xy+2y^2,\ f_{xx}=6x+4y$ ， $f_{xy}=4x+4y$

13. $f_y=e^{-xy}(-x)\dfrac{\sin x}{x}=-e^{-xy}\sin x$

$$f_{yx}=-(-ye^{-xy}\sin x+e^{-xy}\cos x)=e^{-xy}(y\sin x-\cos x)$$

$$f_{xy}=f_{yx}=e^{-xy}(y\sin x-\cos x)$$

14. $f_x=e^{yz}+ye^{zx}z+ze^{xy}y=e^{yz}+yze^{zx}+zye^{xy}$

$$f_{xy}=e^{yz}z+ze^{xz}+ze^{xy}+zye^{xy}x=ze^{yz}+ze^{xz}+(1+xy)ze^{xy}\text{ ，}$$

$$f_{yxz}=1e^{yz}+ze^{yz}y+1e^{xz}+ze^{xz}x+(1+xy)e^{xy}=(1+yz)e^{yz}+(1+xz)e^{xz}+(1+xy)e^{xy}$$

$$f_{zxy}=f_{yxz}$$

15. 因 $z=f(x,y)=\sqrt{9-x^2-y^2}$ ， $\dfrac{\partial z}{\partial y}=\dfrac{-y}{\sqrt{9-x^2-y^2}}$

故相交曲線在點 $(2,2,1)$ 之切線斜率為 $\left.\dfrac{\partial z}{\partial y}\right|_{(2,2)}=-2$ ，

且所求之切線方程式為

$$\begin{cases} z-1=-2(y-2) \\ x=2 \end{cases}$$

16. 因 $z=x^2+2xy$ ， $\dfrac{\partial z}{\partial x}=2x+2y$

故相交曲線在點 $(-1,2,-3)$ 之切線斜率為 $\left.\dfrac{\partial z}{\partial x}\right|_{(-1,2)}=2$ ，

且所求之切線方程式為

$$\begin{cases} z+3=2(x+1) \\ y=2 \end{cases}$$

17. 因為 $f(x,y)=\dfrac{1}{2}\ln(x^2+y^2)$ ，

$$f_x=\frac{1}{2}\cdot\frac{2x}{x^2+y^2}=\frac{x}{x^2+y^2}$$

$$f_{xx}=\frac{(x^2+y^2)1-x(2x)}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

同理 $f_{yy}=\dfrac{x^2-y^2}{(x^2+y^2)^2}$

所以 $f_{xx}+f_{yy}=0$

18. 因為 $f(x,y)=(x^2+y^2+z^2)^{-\frac{1}{2}}$ ，

$$f_x=-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)=-x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$f_{xx}=-[1(x^2+y^2+z^2)^{-\frac{3}{2}}+x(-\frac{3}{2})(x^2+y^2+z^2)^{-\frac{5}{2}}2x]$$

$$=(x^2+y^2+z^2)^{-\frac{5}{2}}[-(x^2+y^2+z^2)+3x^2]$$

$$=(x^2+y^2+z^2)^{-\frac{5}{2}}(2x^2-y^2-z^2)$$

同理

$$f_{yy}=(x^2+y^2+z^2)^{-\frac{5}{2}}(2y^2-x^2-z^2),\ f_{zz}=(x^2+y^2+z^2)^{-\frac{5}{2}}(2z^2-x^2-y^2)$$

所以 $f_{xx}+f_{yy}+f_{zz}=0$

習題 9-4

1. $$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$=(6xy+2)2t+(3x^2+1)3e^{4t}4$$

$$=4(3xy+1)t+12(3x^2+1)e^{4t}$$

2. $$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$=(y+z)(-\frac{1}{t^2})+(x+z)e^t+(y+x)e^{-t}(-1)$$

$$=-\frac{1}{t^2}(y+z)+(x+z)e^t-(y+x)e^{-t}$$

3.

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=2xe^t+2yte^s,$$

$$\frac{\partial u}{\partial t}==\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=2xse^t+2ye^s$$

4. $$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial s}=(1+y^2z)2s+(z^2+2xyz)t+(2yz+xy^2)2s$$

$$=2(1+y^2z)s+(z^2+2xyz)t+2(2yz+xy^2)s$$ ，

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial t}=(1+y^2z)2t+(z^2+2xyz)s+(2yz+xy^2)(-2t)$$

$$=2(1+y^2z)t+(z^2+2xyz)s-2(2yz+xy^2)t$$

5. 證(1) 因為

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta,$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}r\cos\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)^2+\frac{1}{r^2}\left(-r\frac{\partial z}{\partial x}\sin\theta+r\frac{\partial z}{\partial y}\cos\theta\right)^2$$

$$=\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)^2+\left(-\frac{\partial z}{\partial x}\sin\theta+\frac{\partial z}{\partial y}\cos\theta\right)^2$$

$$=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+2\frac{\partial z}{\partial x}\cos\theta\frac{\partial z}{\partial y}\sin\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta+\left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta-2\frac{\partial z}{\partial x}\sin\theta\frac{\partial z}{\partial y}\cos\theta+\left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta$$

$$=\left(\frac{\partial z}{\partial x}\right)^2(\cos^2\theta+\sin^2\theta)+\left(\frac{\partial z}{\partial y}\right)^2(\cos^2\theta+\sin^2\theta)$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

證(2)

$$\frac{\partial^2 z}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)=\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)$$

$$=\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial r}\right)\cos\theta+\left(\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right)\sin\theta$$

$$=\left(\frac{\partial^2 z}{\partial x^2}\cos\theta+\frac{\partial^2 z}{\partial y\partial x}\sin\theta\right)\cos\theta+\left(\frac{\partial^2 z}{\partial x\partial y}\cos\theta+\frac{\partial^2 z}{\partial y^2}\sin\theta\right)\sin\theta$$

$$=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+2\frac{\partial^2 z}{\partial y\partial x}\sin\theta\cos\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta$$

$$\frac{\partial^2 z}{\partial \theta^2}=\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial \theta}\right)=\frac{\partial}{\partial \theta}\left(-r\frac{\partial z}{\partial x}\sin\theta+\frac{\partial z}{\partial y}r\cos\theta\right)$$

$$=-r\left[\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial \theta}+\frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial \theta}\right)\sin\theta+\frac{\partial z}{\partial x}\cos\theta\right]+r\left[\left(\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial \theta}+\frac{\partial^2 z}{\partial^2 y}\frac{\partial y}{\partial \theta}\right)\cos\theta+\frac{\partial z}{\partial y}(-\sin\theta)\right]$$

$$=-r\left[\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta)+\frac{\partial^2 z}{\partial y\partial x}r\cos\theta\right)\sin\theta+\frac{\partial z}{\partial x}\cos\theta\right]$$

$$+r\left[\left(\frac{\partial^2 z}{\partial x\partial y}(-r\sin\theta)+\frac{\partial^2 z}{\partial^2 y}r\cos\theta\right)\cos\theta+\frac{\partial z}{\partial y}(-\sin\theta)\right]$$

$$=r^2\left(\frac{\partial^2 z}{\partial x^2}\sin^2\theta-2\frac{\partial^2 z}{\partial y\partial x}\sin\theta\cos\theta+\frac{\partial^2 z}{\partial^2 y}\cos^2\theta\right)-r\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)$$

所以

$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+2\frac{\partial^2 z}{\partial y\partial x}\sin\theta\cos\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta+\frac{1}{r}[\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta]$$

$$+\frac{1}{r^2}[r^2(\frac{\partial^2 z}{\partial x^2}\sin^2\theta-2\frac{\partial^2 z}{\partial y\partial x}\sin\theta\cos\theta+\frac{\partial^2 z}{\partial^2 y}\cos^2\theta)-r(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta)]$$

$$=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$$

6. $\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}$

$$=\frac{1}{1+(\frac{x}{y})^2}\cdot\frac{1}{y}+\frac{1}{1+(\frac{x}{y})^2}\cdot(-\frac{x}{y^2})e^x$$

$$=\frac{y}{y^2+x^2}-\frac{xe^x}{y^2+x^2}$$

$$=\frac{y-xe^x}{y^2+x^2}$$

7. $\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}+\frac{\partial u}{\partial z}\frac{dz}{dx}$

$$=\frac{2x}{2\sqrt{x^2+y^2+z^2}}+\frac{2y}{2\sqrt{x^2+y^2+z^2}}(\sin x+x\cos x)+\frac{2z}{2\sqrt{x^2+y^2+z^2}}\sec^2 x$$

$$=\frac{x+y\sin x+xy\cos x+z\sec^2 x}{\sqrt{x^2+y^2+z^2}}$$

8.因為 $z=\frac{xy}{x^2+y^2}=\frac{r\cos\theta r\sin\theta}{r^2}=\cos\theta\sin\theta=\frac{1}{2}\sin 2\theta$ ，所以

$$\frac{\partial z}{\partial r}=0,\ \frac{\partial z}{\partial \theta}=\cos 2\theta$$

則

$$\frac{\partial z}{\partial r}\Big|_{r=3,\theta=\pi/6}=0$$

$$\frac{\partial z}{\partial \theta}\Big|_{r=3,\theta=\pi/6}=\frac{1}{2},$$

9. 證： 令 $u=x^2+y^2,\Rightarrow z=f(u)$

$$\frac{\partial z}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}=\frac{df}{du}2x,\qquad (1)$$

$$\frac{\partial z}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}=\frac{df}{du}2y,\qquad (2)$$

$x\times(2)-y\times(1)$ 得

$$x\frac{\partial z}{\partial y}-y\frac{\partial z}{\partial x}=x\frac{df}{du}2y-y\frac{df}{du}2x=$$

10. 證： 令 $u=y+ax, \Rightarrow F=f(u)$

$$\frac{\partial F}{\partial x}=\frac{dF}{du}\frac{\partial u}{\partial x}=\frac{df}{du}a$$

$$\frac{\partial^2 F}{\partial x^2}=\frac{\partial}{\partial x}(a\frac{df}{du})=a\frac{d^2 f}{du^2}\frac{\partial u}{\partial x}=a\frac{d^2 f}{du^2}a=a^2\frac{d^2 f}{du^2} \quad (1)$$

$$\frac{\partial F}{\partial y}=\frac{dF}{du}\frac{\partial u}{\partial y}=\frac{df}{du}1$$

$$\frac{\partial^2 F}{\partial y^2}=\frac{\partial}{\partial y}(\frac{df}{du})=\frac{d^2 f}{du^2}\frac{\partial u}{\partial y}=\frac{d^2 f}{du^2}1=\frac{d^2 f}{du^2}$$

由(1)(2)得證 $\frac{\partial^2 F}{\partial x^2}=a^2\frac{\partial^2 F}{\partial y^2}$

11. 設時間為 t 秒時二股長分別為 x 吋、 y 吋，斜邊長 r 吋，則

$r^2=x^2+y^2$

二邊對 t 微分得

$2r\frac{dr}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$ 或 $r\frac{dr}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}$

當 $x=3, y=4$ 時 $r=5$ ，所以

$$5\frac{dr}{dt}=3(10)+4(10)$$

$\Rightarrow \frac{dr}{dt}=14$ 吋/秒

12. 設時間為 t 秒時長, 寬, 高分別為 x,y,z 呎, 已知

$$\frac{dx}{dt}=\frac{dy}{dt}=5, \frac{dz}{dt}=-8$$

長方體體積 $V=xyz$ ，則

$$\frac{dV}{dt}=\frac{\partial V}{\partial x}\frac{dx}{dt}+\frac{\partial V}{\partial y}\frac{dy}{dt}+\frac{\partial V}{\partial z}\frac{dz}{dt}=yz(5)+xz(5)+xy(-8)$$

$$\left.\frac{dV}{dt}\right|_{x=3,y=6,z=10}=6\cdot 10\cdot 5+3\cdot 10\cdot 5+3\cdot 6\cdot(-8)=306$$

所以體積以306呎 ^{3}/秒的速度增加。

13. 令 $F(x,y)=x^2+3xy-2y^2-5=0$

➔ $\frac{dy}{dx}=-\frac{\partial F/\partial x}{\partial F/\partial y}=-\frac{2x+3y}{3x-4y}$

14. 令 $F=x^2e^{3y}+y^3-1$

➔ $\frac{dy}{dx}=-\frac{\partial F/\partial x}{\partial F/\partial y}=-\frac{2xe^{3y}}{x^2e^{3y}3+3y^2}=-\frac{2xe^{3y}}{3(x^2e^{3y}+y^2)}$

15. 令 $F=xy^2+yz^2+x^3+z^3-8$

➔ $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{y^2+3x^2}{2yz+3z^2}$

$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{2xy+z^2}{2yz+3z^2}$

$$\Rightarrow \frac{\partial z}{\partial x}\Big|_{(0,0,2)} = 0, \frac{\partial z}{\partial y}\Big|_{(0,0,2)} = \frac{1}{3}$$

16. 令 $F = xe^{yz} + ye^{xz} - y^2 + 3$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{e^{yz} + yze^{xy}}{xye^{yz} + xye^{xz}},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{xze^{yz} + e^{xz} - 2y}{xye^{yz} + xye^{xz}}$$

17. 將二方程式對 x 作微分，得

$$y + x\frac{dy}{dx} + \frac{dy}{dx}z + y\frac{dz}{dx} + z + x\frac{dz}{dx} = 0$$

$$1 - \frac{dy}{dx} - \frac{dz}{dx} = 0$$

則

$$(x+z)\frac{dy}{dx} + (x+y)\frac{dz}{dx} = -(y+z)$$

$$\frac{dy}{dx} + \frac{dz}{dx} = 1$$

對此聯立方程式求解，可得

$$\frac{dy}{dx} = \frac{\begin{vmatrix} -(y+z) & x+y \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} x+z & x+y \\ 1 & 1 \end{vmatrix}} = \frac{-(x+2y+z)}{z-y}$$

$$\frac{dy}{dx} = \frac{\begin{vmatrix} x+z & -(y+z) \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} x+z & x+y \\ 1 & 1 \end{vmatrix}} = \frac{x+y+2z}{z-y}$$

18. 將二方程式對 x 作微分，得

$$y^2 + x2y\frac{dy}{dx} + \frac{dz}{dx} + 2 = 0$$

$$z + x\frac{dz}{dx} - 2y\frac{dy}{dx} = 0$$

則

$$2xy\frac{dy}{dx} + \frac{dz}{dx} = -(y^2 + 2)$$

$$-2y\frac{dy}{dx} + x\frac{dz}{dx} = -z$$

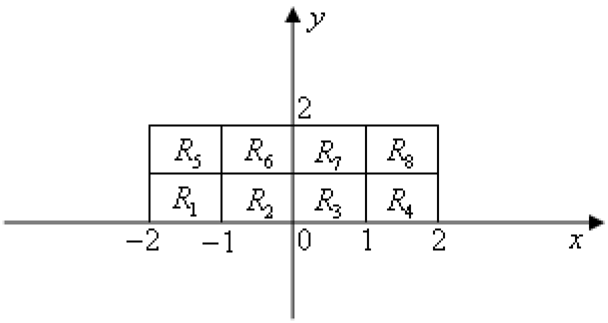
$$\frac{dy}{dx}=\frac{\begin{vmatrix}-(y^2+2) & 1\\ -z & x\end{vmatrix}}{\begin{vmatrix}2xy & 1\\ -2y & x\end{vmatrix}}=\frac{-xy^2-2x+z}{2x^2y+2y}$$

$$\frac{dz}{dx}=\frac{\begin{vmatrix}2xy & -(y^2+2)\\ -2y & -z\end{vmatrix}}{\begin{vmatrix}2xy & 1\\ -2y & x\end{vmatrix}}=\frac{-(xyz+2y^3+4y)}{2x^2y+2y}$$

習題 10-1

1. 令 $f(x,y)=4xy$，將 R 分割成 $R_1,R_2,\cdots,R_8$（如下圖），並取每子區域的中心點 (x_i,y_i) 對應作高，則 $f(x_1,y_1)=f(-\frac{3}{2},\frac{1}{2})=-3$， $f(x_2,y_2)=f(-\frac{1}{2},\frac{1}{2})=-1$， $f(x_3,y_3)=f(\frac{1}{2},\frac{1}{2})=1$， $f(x_4,y_4)=f(\frac{3}{2},\frac{1}{2})=3,\ f(x_5,y_5)=f(-\frac{3}{2},\frac{3}{2})=-9,$ $f(x_6,y_6)=f(-\frac{1}{2},\frac{3}{2})=-3,\ f(x_7,y_7)=f(\frac{1}{2},\frac{3}{2})=3,\ f(x_8,y_8)=f(\frac{3}{2},\frac{3}{2})=9$，所以

$$\iint_R 4xydA\approx\sum_{i=1}^{8}f(x_i,y_i)\Delta A_i=1\cdot\sum_{i=1}^{8}f(x_i,y_i)=0$$



2. $\iint_R 5dA=5|R|=5\times 2=10$

習題 10-2

1. $\iint_S (x^2+y^2)dA=\int_0^1\int_0^2(x^2+y^2)dxdy$

$$=\int_0^1 \frac{1}{3}x^3+y^2x\Big|_0^2 dy$$

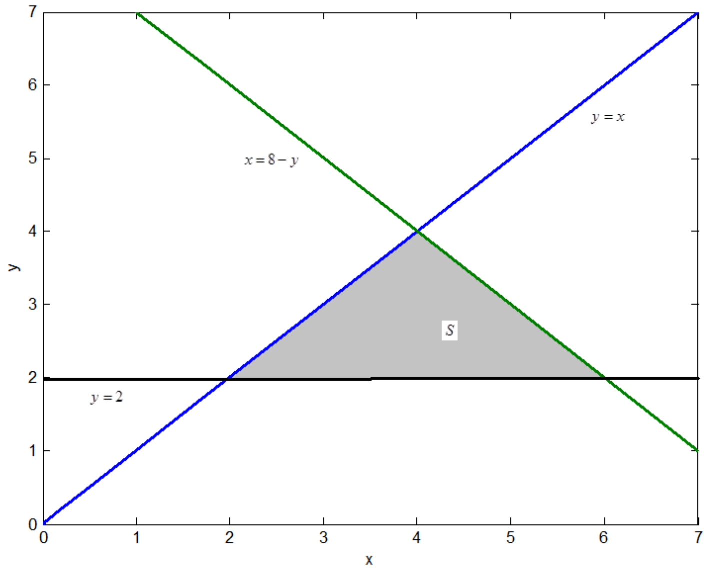
$$=\int_0^1 \frac{8}{3}+2y^2dy$$

$$=\frac{8}{3}y+\frac{2}{3}y^3\Big|_0^1=\frac{10}{3}$$

2. $\iint_S y\,dA = \int_2^4 \int_y^{8-y} y\,dx\,dy = \int_2^4 yx\Big|_y^{8-y} dy$

$$= \int_2^4 y(8-y-y)dy$$

$$= 4y^2 - \frac{2}{3}y^3\Big|_2^4 = \frac{32}{3}$$

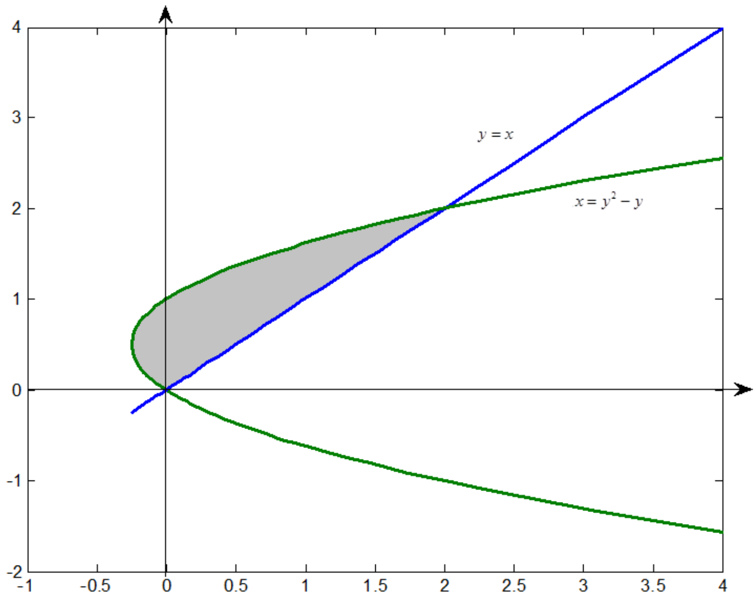


3. $\iint_S 2x\,dA = \int_0^2 \int_{y^2-y}^{y} 2x\,dx\,dy$

$$= \int_0^2 x^2\Big|_{y^2-y}^{y} dy$$

$$= \int_0^2 y^2 - (y^2-y)^2 dy$$

$$= \int_0^2 -y^4 + 2y^3 dy = -\frac{1}{5}y^5 + \frac{2}{4}y^4\Big|_0^2 = \frac{8}{5}$$

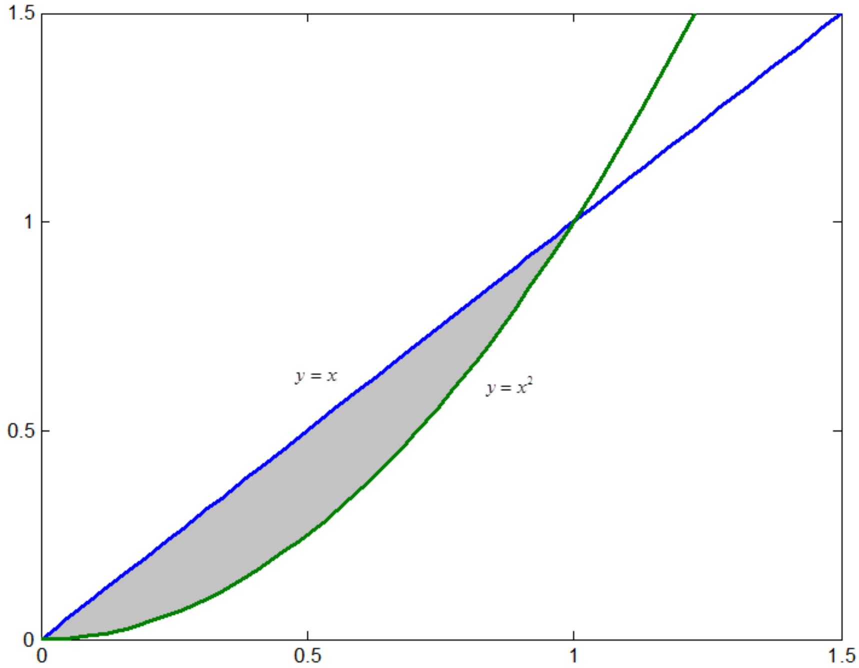


4. $$\iint_S 2xy\,dA = \int_0^1\int_{x^2}^{x} 2xy\,dy\,dx$$

$$= \int_0^1 xy^2\Big|_{x^2}^{x} dx$$

$$= \int_0^1 x(x^2 - x^4)dx$$

$$= \frac{1}{4}x^4 - \frac{1}{6}x^6\Big|_0^1 = \frac{1}{12}$$



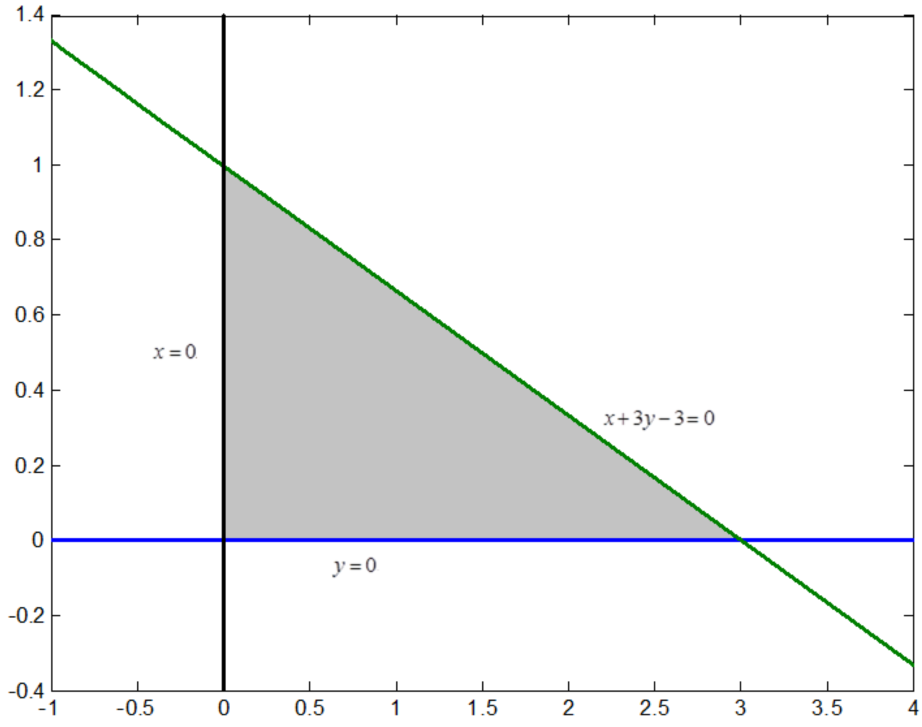
5. $$\iint_S (3x-y)\,dA = \int_0^1\int_0^{3-3y}(3x-y)\,dx\,dy$$

$$= \int_0^1 \frac{3}{2}x^2 - yx\Big|_0^{3-3y} dy$$

$$= \int_0^1 \frac{3}{2}(3-3y)^2 - y(3-3y)dy$$

$$= \int_0^1 \left(\frac{33}{2}y^2 - 30y + \frac{27}{2}\right)dy$$

$$= \frac{33}{6}y^3 - 15y^2 + \frac{27}{2}y\Big|_0^1 = 4$$



6. $\int_0^1\int_0^{y^2} 2ye^x dxdy = \int_0^1 2y\, e^x\Big|_0^{y^2} dy$

$= \int_0^1 2y(e^{y^2}-1)dy$ （令 $u=y^2, du=2ydy$ ）

$= \int_0^1 (e^u-1)du$

$= e^u - u\Big|_0^1 = (e-1)-1 = e-2$

7. $\int_0^2\int_{-1}^1 \sqrt{|y-x^2|}\, dxdy = \int_{-1}^1\int_0^{x^2} \sqrt{x^2-y}\, dydx + \int_{-1}^1\int_{x^2}^2 \sqrt{y-x^2}\, dydx$

$= -\frac{2}{3}\int_{-1}^1 (x^2-y)^{\frac{3}{2}}\Big|_0^{x^2} dx + \frac{2}{3}\int_{-1}^1 (y-x^2)^{\frac{3}{2}}\Big|_{x^2}^2 dx$

$= -\frac{2}{3}\int_{-1}^1 (0-x^3)dx + \frac{2}{3}\int_{-1}^1 (2-x^2)^{\frac{3}{2}} dx$

$= \frac{2}{3}\cdot\frac{1}{4}x^4\Big|_{-1}^1 + \frac{4}{3}\int_0^1 (2-x^2)^{\frac{3}{2}} dx$ （令 $u=\sqrt{2}\sin\theta, du=\sqrt{2}\cos\theta d\theta$ ）

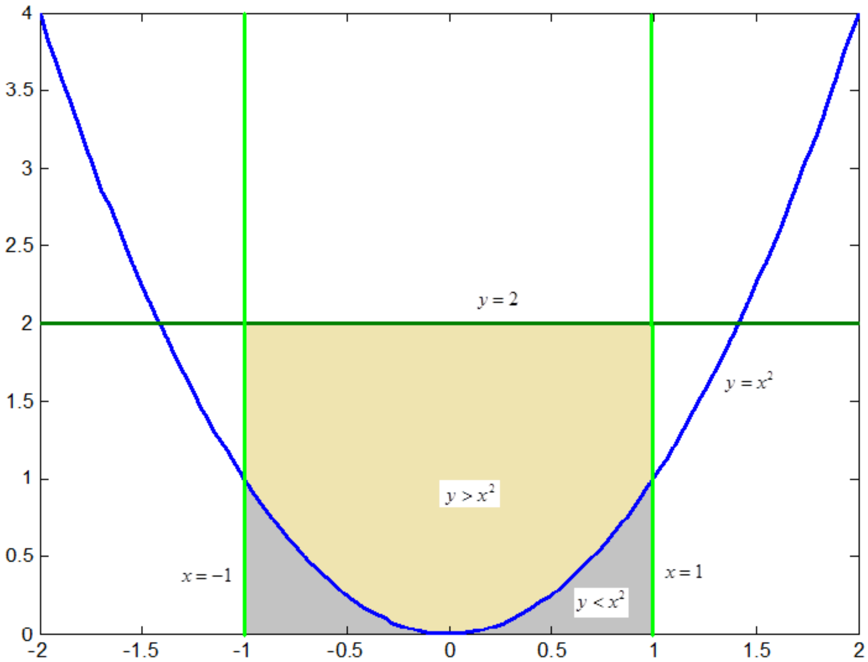
$= 0 + \frac{16}{3}\int_0^{\frac{\pi}{4}} \cos^4\theta d\theta$

$= \frac{16}{3}\int_0^{\frac{\pi}{4}} (\frac{1+\cos 2\theta}{2})^2 d\theta$

$$=\frac{4}{3}\int_0^{\frac{\pi}{4}}(1+2\cos 2\theta+\cos^2 2\theta)d\theta$$

$$=\frac{4}{3}\int_0^{\frac{\pi}{4}}(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2})d\theta$$

$$=\frac{4}{3}[\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta]\Big|_0^{\frac{\pi}{4}}=\frac{8+3\pi}{6}$$



8. $$\int_0^1\int_0^1\frac{x-y}{(x+y)^3}\,dx\,dy=\int_0^1\int_0^1\frac{x+y-2y}{(x+y)^3}\,dx\,dy$$

$$=\int_0^1\int_0^1\frac{1}{(x+y)^2}+\frac{-2y}{(x+y)^3}\,dx\,dy$$

$$=\int_0^1-\frac{1}{x+y}+\frac{y}{(x+y)^2}\Big|_0^1\,dy$$

$$=\int_0^1-\frac{1}{1+y}+\frac{y}{(1+y)^2}\,dy$$

$$=\int_0^1-\frac{1}{1+y}+\frac{y+1-1}{(1+y)^2}\,dy$$

$$=\int_0^1 -\frac{1}{(1+y)^2}\,dy$$

$$=\left.\frac{1}{1+y}\right|_0^1=\frac{1}{2}-1=-\frac{1}{2}$$

$$\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx=-\int_0^1\int_0^1 \frac{y-x}{(x+y)^3}\,dy\,dx=-(-\frac{1}{2})=\frac{1}{2}$$

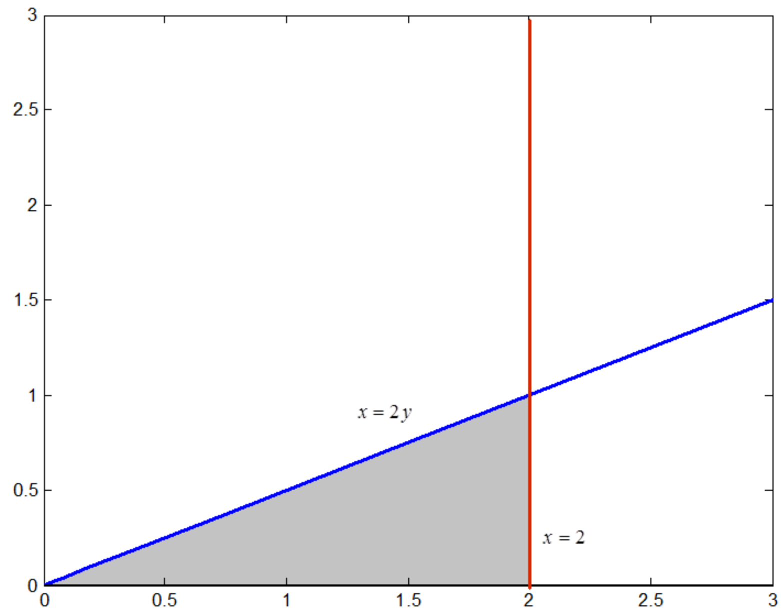
9. $$\int_0^1\int_{2y}^2 e^{x^2}dxdy=\int_0^2\int_0^{\frac{x}{2}} e^{x^2}dydx$$

$$=\int_0^2 e^{x^2}\left.y\right|_0^{\frac{x}{2}}dx$$

$$=\int_0^2 e^{x^2}\frac{x}{2}\,dx$$

$$=\frac{1}{4}\int_0^4 e^u\,du \quad (u=x^2, du=2xdx)$$

$$=\frac{1}{4}\left.e^u\right|_0^4=\frac{1}{4}(e^4-1)$$



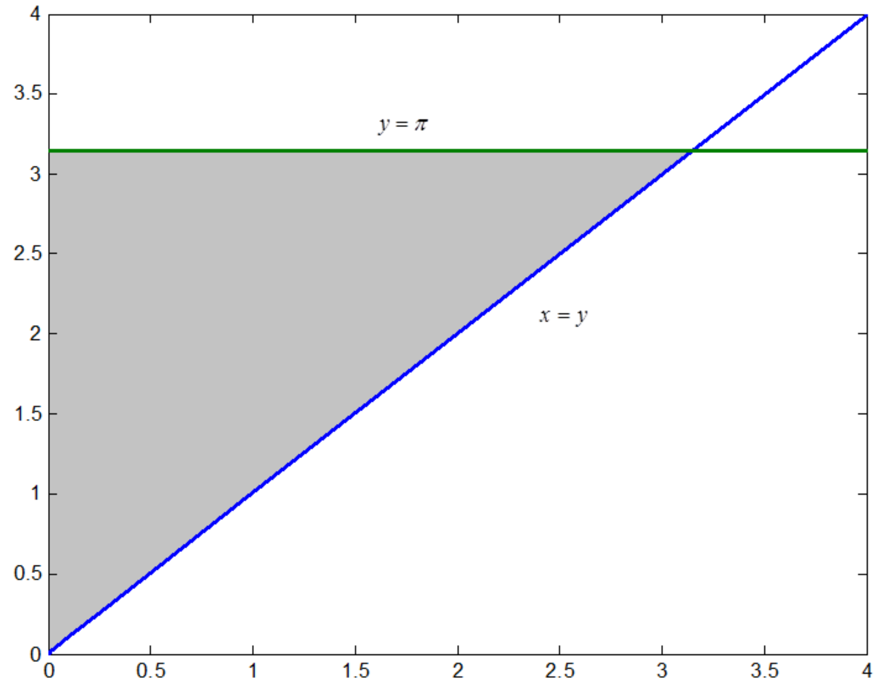
10. $$\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}dydx=\int_0^{\pi}\int_0^{y}\frac{\sin y}{y}dxdy$$

$$= \int_0^{\pi} \frac{\sin y}{y} x\Big|_0^y dy$$

$$= \int_0^{\pi} \frac{\sin y}{y} y\, dy$$

$$= \int_0^{\pi} \sin y\, dy$$

$$= -\cos y\Big|_0^{\pi} =$$



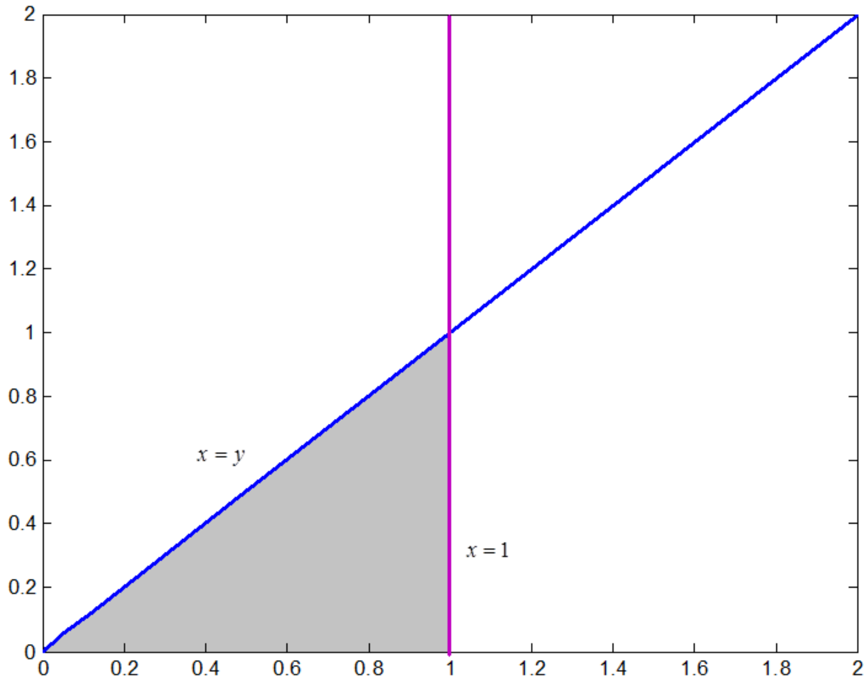
11. $$\int_0^1 \int_y^1 \frac{1}{1+x^4}\, dxdy = \int_0^1 \int_0^x \frac{1}{1+x^4}\, dydx$$

$$= \int_0^1 \frac{1}{1+x^4}\, y\Big|_0^x dx$$

$$= \int_0^1 \frac{1}{1+x^4} x\, dx$$

$$= \frac{1}{2} \int_0^1 \frac{1}{1+u^2} du \quad (u = x^2, du = 2xdx)$$

$$= \frac{1}{2} \tan^{-1} u\Big|_0^1 = \frac{1}{2}(\frac{\pi}{4} - 0) = \frac{\pi}{8}$$



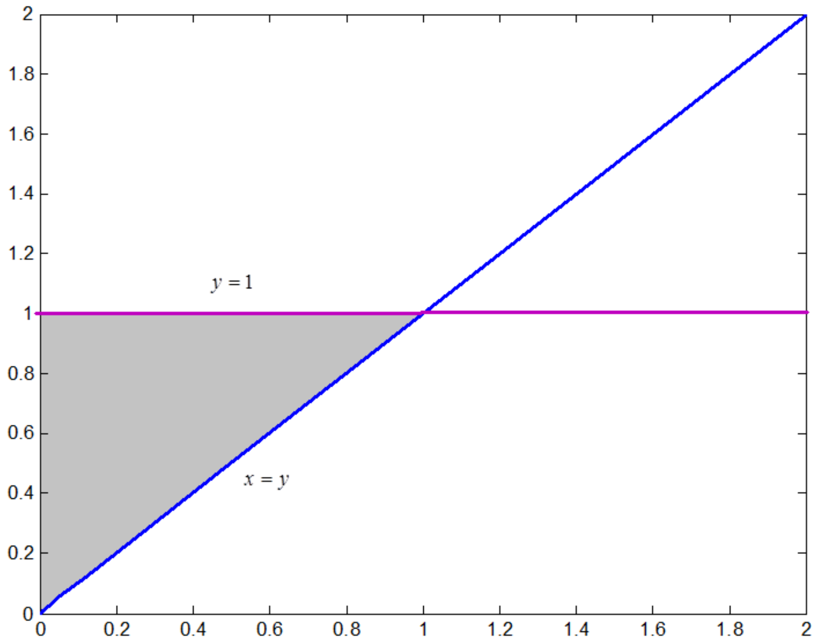
12. $$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \int_0^1 \int_0^y \sin(y^2)\,dx\,dy$$

$$= \int_0^1 \sin(y^2)\,y\,dy$$

$$= \int_0^1 \sin(y^2)\,y\,d \quad (u = y^2, du = 2ydy)$$

$$= \frac{1}{2}\int_0^1 \sin u\,du$$

$$= \frac{1}{2}(-\cos u|_0^1) = \frac{1}{2}(1-\cos 1)$$



y = 1
x = y
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2